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THE SOLUTION OF ALGEBRAIC EQUATIONS BY PARTIAL DIFFERENTIAL EQUATIONS

BY H. A. SAYRE

1. LET the algebraic equation of degree n in z be

$$z^n + nxz + ny = 0.$$

Let the solution of this equation be $z = F(x, y)$. With the usual notation for partial derivatives with respect to x and y, we have

$$p = -\frac{z}{z^{n-1} + x}, \qquad q = -\frac{1}{z^{n-1} + x},$$

$$r = \frac{2zx - (n-3)z^n}{(z^{n-1} + x)^3}, \qquad s = \frac{x - (n-2)z^{n-1}}{(z^{n-1} + x)^3},$$

$$t = -\frac{(n-1)z^{n-2}}{(z^{n-1} + x)^3},$$

$$a = \frac{(12n - 2n^2 - 16)z^{2n-1} + z^n x(n^2 + 6n - 19) - 6zx^2}{(z^{n-1} + x)^5},$$

$$\beta = \frac{(9n - 2n^2 - 9)z^{2n-2} + (n^2 + 3n - 8)z^{n-1}x - 2x^2}{(z^{n-1} + x)^5},$$

$$\gamma = \frac{(n-1)z^{n-2}\{(4 - 2n)z^{n-1} + (n+1)x\}}{(z^{n-1} + x)^5},$$

$$\delta = \frac{(n-1)(1-2n)z^{2n-4} + (n-1)(n-2)xz^{n-3}}{(z^{n-1} + x)^5}.$$

Among these derivatives there exist relations such as

$$p - qz = 0,$$

$$rt - s^2 - pqt + q^2 s = 0,$$

$$pz^{n-2} + qx + 1 = 0,$$

$$z = (n-1)\,px + nqy.$$

When n is assigned it may be possible to form a differential equation, into whose solution z and the arbitrary functions enter in such a way, that when the algebraic equation is considered as a particular solution, the integral equation reduces to

$$z - F(x, y) = 0,$$

where F is a known function. If so, the algebraic equation is solved.

2. Solution of the Quadratic. For $n = 2, pz^{n-2} + qx + 1 = 0$, reduces to

$$p + qx + 1 = 0.$$

Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{x} = \frac{dz}{-1}.$$

Two integrals are

$$x^2 - 2y = \text{constant}, \qquad z + x = \text{constant};$$

hence the general integral is

$$z = -x + \phi(x^2 - 2y).$$

ϕ must now be determined so as to make

$$z^2 + 2\,zx + 2y = 0$$

a particular solution.

1°. In the quadratic, when $x = 0$, we get

$$z = \pm\sqrt{-2y}, \text{ hence } \phi(-2y) = \pm\sqrt{-2y},$$

which determines the form of ϕ. The integral equation now reduces to

$$z = -x \pm \sqrt{x^2 - 2y},$$

which is the solution of the quadratic.

2°. Substituting $z = -x + \phi$ in the quadratic, we get $\phi^2 = x^2 - 2y$. This is the well known solution in which the second term is removed by a linear substitution.

3°. The integral equation suggests changing the variables from x and y

to x and v, where $v = x^2 - 2y$. The quadratic becomes

$$z^2 + 2zx + x^2 = v, \text{ hence } z + x = \pm \sqrt{v}.$$

This is the common method of completing the square.

Among the differential equations which may be used for solving the quadratic are the following:

1. $q = \dfrac{1}{\sqrt{x^2 - 2y}}$, $\quad z = \psi(x) + \sqrt{x^2 - 2y}$.
2. $rx + sx^2 - p - 1 = 0$, $\quad z = -x + \phi(x^2 - 2y) + \psi(y)$.
3. $s + tx = 0$, $\quad z = \psi(x) + \phi(x^2 - 2y)$.
4. $\beta x + \gamma x^2 - s = 0$, $\quad z = \phi(x) + \psi(x^2 - 2y) + \theta(y)$.

3. Solution of the Cubic. For the cubic we have $r + tx = 0$. When $n = 3$, $z = (n - 1)\, px + nqy$ becomes

$$z = 2px + 3qy,$$

whence

$$p = 2rx + 2v + 3sy.$$

Multiplying $r + tx = 0$ by x, and subtracting from the preceding equation, we get

$$rx + 3sy - tx^2 + p = 0.$$

To integrate this equation let us use Monge's method. The subsidiary equations are

$$\frac{dy}{dx} = \frac{3y \pm \sqrt{9y^2 + 4x^3}}{2x},$$

$$2d(px) - \frac{4x^3}{3y \pm \sqrt{9y^2 + 4x^3}}\, dq = 0,$$

from which we have

$$-3y + \sqrt{9y^2 + 4x^3} = c_1,$$

$$2px - (-3y + \sqrt{9y^2 + 4x^3})q = c_2,$$

$$-3y - \sqrt{9y^2 + 4x^3} = c_3,$$

$$2px - (-3y - \sqrt{9y^2 + 4x^3})q = c_4.$$

The intermediate integrals are

$$2px - (-3y + \sqrt{9y^2 + 4x^3})\, q = \theta_1\, (-3y + \sqrt{9y^2 + 4x^3}),$$

$$2px - (-3y - \sqrt{9y^2 + 4x^3})\, q = \theta_2\, (-3y - \sqrt{9y^2 + 4x^3}).$$

Putting $u = -3y,\ v = \sqrt{9y^2 + 4x^3})$, we get

$$dz = \frac{\{ [u+v]\, \theta_2(u-v) - [u-v]\, \theta_1(u+v) \}}{4xv}\, dx + \frac{\theta_2(u-v) - \theta_1(u+v)}{2v}\, dy,$$

or

$$dz = \frac{\{ [u+v]\, \theta_2(u-v) - (u-v)\, \theta_1(u+v) \}\, (vdv - udu)}{6v(v^2 - u^2)} - \frac{\theta_2(u-v) - \theta_1(u+v)}{6v}\, du$$

$$= \frac{\theta_2(u-v)}{6\,(u-v)}\, d(u-v) + \frac{\theta_1(u+v)}{6\,(u+v)}\, d(u+v).$$

The integral is

$$z = \phi(u+v) + \psi(u-v)$$

$$= \phi(-3y + \sqrt{9y^2 + 4x^3}) + \psi(-3y - \sqrt{9y^2 + 4x^3}),$$

where ϕ and ψ are arbitrary functions.

Putting $-3y + \sqrt{9y^2 + 4x^3} = 2u_1$, $-3y - \sqrt{9y^2 + 4x^3} = 2v_1$, the general integral becomes

$$z = \phi_1(u_1) + \psi_1(v_1),$$

while the cubic transforms into

$$z^3 - 3z\sqrt[3]{u_1 v_1} - (u_1 + v_1) = 0.$$

The initial values for the cubic are

u_1	v_1	z
0	0	0
0	v_1	$\sqrt[3]{v_1}$
u_1	0	$\sqrt[3]{u_1}$.

Substituting these values in the general integral,

$$\begin{aligned} 0 &= \phi_1(0) + \psi_1(0), \\ \sqrt[3]{v_1} &= \phi_1(0) + \psi_1(v_1), \\ \sqrt[3]{u_1} &= \phi_1(u_1) + \psi_1(0). \end{aligned}$$

By addition

$$\sqrt[3]{u_1} + \sqrt[3]{v_1} = \phi_1(u_1) + \psi_1(v_1)\,;$$

hence

$$z = \sqrt[3]{u_1} + \sqrt[3]{v_1}\,;$$

restoring the original variables,

$$z = \sqrt[3]{\frac{-3y + \sqrt{9y^2 + 4x^3}}{2}} + \sqrt[3]{\frac{-3y - \sqrt{9y^2 + 4x^3}}{2}}.$$

Since $\sqrt[3]{u_1\, v_1} = -x$, the admissible roots are

$$\begin{aligned} z_1 &= \sqrt[3]{u_1} + \sqrt[3]{v_1}, \\ z_2 &= w\sqrt[3]{u_1} + w^2\sqrt[3]{v_1}, \\ z_3 &= w^2\sqrt[3]{u_1} + w\sqrt[3]{v_1}, \end{aligned}$$

where $1 + w + w^2 = 0$.

4. Solution of the Biquadratic. For the biquadratic

$$z = 3vx + 4qy, \qquad p = 3rx + 3p + 4sy.$$

Put $A = 3rx + 4sy + 2p = 0$, $B = 3ax + 4\,\beta y + 5r = 0$, $C = \alpha + \delta x = 0$. Then

$$Bx - Cx^2 - A = 2\alpha x^2 - \delta x^3 + 4\beta xy + 2rx - 4sy - 2p = 0$$

is the equation that will be considered. Changing the variables from x and y to ξ_1 and ξ_2, the coefficients of $\frac{\delta^3 z}{\delta \xi_1^3}$ and $\frac{\delta^3 z}{\delta \xi_2^3}$ will vanish if

$$\frac{\partial \xi_1}{dy} = m_1 \frac{\partial \xi_1}{\partial x}, \qquad \frac{\partial \xi_2}{dy} = m_2 \frac{\partial \xi_2}{\partial x},$$

where m_1 and m_2 are roots of the cubic equation

$$\lambda^3 - \frac{4y}{x^2}\lambda - \frac{2}{x} = 0.$$

$$m_1 = \frac{2}{x}\left[\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + \sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}\right],$$

$$m_2 = \frac{2}{x}\left[w\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + w^2\sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}\right].$$

Using the simplest values for ξ_1 and ξ_2, we have

$$\xi_1 = \sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + \sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}},$$

$$\xi_2 = w\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + w^2\sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}.$$

The differential equation now becomes

$$\frac{\partial^3 z}{\partial \xi_1^2 \partial \xi_2} - \frac{\partial^3 z}{\partial \xi_1 \partial \xi_2^2} = 0,$$

and has for its solution

$$z = \phi(\xi_1) + \psi(\xi_2) + \theta(\xi_3),$$

where ξ_1, ξ_2, ξ_3 are roots of the cubic equation

$$\xi^3 - y\xi - \frac{x^2}{4} = 0.$$

The biquadratic may be written

$$z^4 + 8z\epsilon\sqrt{\xi_1\xi_2\xi_3} - 4(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) = 0,$$

where $\epsilon^2 = 1$ and ϵ has the sign of x.

Permuting ξ_1, ξ_2, ξ_3 leaves the biquadratic unchanged, hence

$$z = \phi(\xi_1) + \phi(\xi_2) + \phi(\xi_3).$$

Substituting this value of z in the biquadratic, we have

$$z^4 - 2\,[\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3)]\, z^2 - 8\phi(\xi_1)\,\phi(\xi_2)\,\phi(\xi_3)\, z$$
$$+ [\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3)]^2$$
$$- 4[\phi^2(\xi_1)\phi^2(\xi_2) + \phi^2(\xi_2)\phi^2(\xi_3) + \phi^2(\xi_3)\phi^2(\xi_1)] = 0.$$

Identifying this equation with the biquadratic,

$$\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3) = 0,$$
$$\phi(\xi_1)\,\phi(\xi_2)\,\phi(\xi_3) = -\,\epsilon\,\sqrt{\xi_1\xi_2\xi_3},$$
$$\phi^2(\xi_1)\,\phi^2(\xi_2) + \phi^2(\xi_2)\,\phi^2(\xi_3) + \phi^2(\xi_3)\,\phi^2(\xi_1) = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1,$$

together with $\xi_1 + \xi_2 + \xi_3 = 0.$
Hence

$$\phi(\xi_1) = \epsilon_1\sqrt{\xi_1},\ \phi(\xi_2) = \epsilon_2\sqrt{\xi_2},\ \phi(\xi_3) = \epsilon_3\sqrt{\xi_3},$$

where $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$, and $\epsilon_1\epsilon_2\epsilon_3 = -\,\epsilon$.

The solution of the biquadratic is therefore

$$z = \epsilon_1\sqrt{\xi_1} + \epsilon_2\sqrt{\xi_2} + \epsilon_3\sqrt{\xi_3}.$$

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